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Substrate Thermo-Refractive Noise for future cryogenic gravitational wave detector

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Abstract

This work reports about the substrate thermo-refractive noise contribution for the future ET interferometer. In this note, the results for sapphire and silicon substrates at low temperatures are discussed.

1. Substrate Thermo-Refractive Noise: literatures

Two references can be mentioned for the substrate thermo-refractive noise formula. The first one was the formula given by S. Rao in his thesis [1]. Unfortunately, this formula is neither demonstrated nor referenced. An other formula was given by Braginsky et al. in 2004 [2]. This second formula is well detailed in the paper and thus, has been chosen for simulations of this note. The expression is:

$$\varphi_n^2 = (kl\beta)^2 \frac{4k_b T^2 \kappa}{(\rho C)^2 l_c} \int_0^\infty \frac{k_\perp dk_\perp}{2\pi} e^{-\frac{w^2 k_\perp^2}{2}} \frac{k_\perp^2}{(2\pi f)^2 + a^4 k^4} \qquad (1.1)$$

With $k = \frac{2\pi}{\lambda}$, $a^2 = \frac{\kappa}{\rho C}$ and $k_\perp^2 = k_x^2 + k_y^2$

l_c is the thickness of the input mirror

β is the thermo-optic coefficient of the substrate material

T is the temperature

k_b is the Boltzmann's constant

κ is the thermal conductivity

C is the specific heat

ρ is the density of the substrate material

r is the size of the beam (intensity of beam decreases as e^{-1} at distance r from center)

This formula can be simplified by the expression:

$$\varphi_n{}^2 = \frac{4\beta^2 k^2 l_c T^2 k_b \kappa}{(\rho C)^2 \pi r^4 (2\pi f)^2} \text{ for adiabatic case, i.e., for } 2\pi f >> \frac{\kappa\omega^2}{\rho C} \qquad (1.2)$$

In the paper of Braginsky, the thermo refractive noise is demonstrated for a type of cavity based on 2 prisms. In the following section, we propose to discuss the formula in order to be adapted for the cavities of the future Einstein Telescope.

2. Comparison to the sensitivity curve

The thermo-refractive noise described in the previous part is generated inside transmissive optics such as the input mirror of the arm cavities or the beamsplitter. This noise is not equivalent to the displacement of a mirror of the arm cavities and as such can not be directly compared directly h, the sensitivity. In this section, we will derive a simple model to convert the thermo-refractive phase noise from the input mirror of the arm cavity to a displacement noise.

We now consider the thermo-refractive noise generated inside the input mirror substrate. It is interesting to note that this noise taken in reflection (so coming toward the central part of the interferometer) will also see the low pass filtering effect of the arm cavity. That has been shown also by Hild in [3] in the case of a full interferometer, LMA performs other simulations for confirmation. So the noise from the arm cavity can be written as:

$$\Delta\varphi_{TR} = 2\varphi_n \frac{1}{\sqrt{1+(f/fc)^2}}$$

With fc the cavity cutoff frequency. Meanwhile, the phase change due the GW signal exiting the arm cavity can be written as:

$\Delta\varphi_{GW} = TF(f)hL$ with h the strain sensitivity in $\sqrt{Hz}$ and L is the length of the interferometer arms. The factor TF(f) is dependent on the finesse of the cavity F, the wavelength of the light λ and the cavity cutoff frequency fc.

$$TF(f) = \frac{4\pi}{\lambda}\frac{2F}{\pi}\frac{1}{\sqrt{1+(f/fc)^2}} \qquad (2.2)$$

Finally, the power and signal recycling cavities are not taken into account in the formula because it takes place in the both phase fluctuations $\Delta\varphi_{GW}$ and $\Delta\varphi_{TR}$ and is thus cancelled.

In order to get a signal:

$$\Delta\varphi_{GW} = \Delta\varphi_{TR} \quad (2.5)$$

In 2009, Benthem and Levin [4] present a new estimation of the substrate thermo-refractive noise which takes into account the beam's elliptical profile and the fact that a standing electromagnetic wave is formed. In the case of our calculation, the beam remains circular and only the correction concerning the standing wave is considered. The formula of the correction can be expressed as follow:

$$Corr = 1 + \frac{(kn)^2 r^2}{(1+(2kn\sqrt{\kappa/C\rho\omega})^4)} \qquad (2.6)$$

This correction has been taken into account in our calculation. Nevertheless, the effect of the standing wave is negligible and does not change the noise.

Here we have considered only one cavity. Then, because the interferometer is composed of two independent arms, the formula must be multiply by $\sqrt{2}$ as: $\sqrt{(\varphi_1)^2 + (\varphi_2)^2}$. The two arms having the same length and configuration: $\varphi_1 = \varphi_2$

Using the general expression of φ_n, we finally show that strain sensitivity of the thermo refractive noise is equal to:

$$h > 2\sqrt{2}\frac{1}{L}\frac{\lambda}{8F}\sqrt{(kl\beta)^2\frac{4k_bT^2\kappa}{(\rho C)^2 l_c}(1+\frac{(kn)^2 r^2}{(1+(2kn\sqrt{\kappa/C\rho\omega})^4)})\int_0^\infty \frac{k_\perp dk_\perp}{2\pi}e^{-\frac{R_b^2 k_\perp^2}{2}}\frac{k_\perp^2}{\omega^2 + a^4 k^4}} \qquad (2.7)$$

In the next section, at low temperature, the heat diffusion length increases and becomes larger than the laser beam size, hence the adiabatic approximation is no longer valid ($2\pi f << \frac{\kappa\omega^2}{\rho C}$). Therefore, to evaluate the non-adiabatic case, the precise formula 2.7 is taken into account. (2.1)

3. Substrate's mirror properties

In this note, two materials have been considered for the mirror substrates: sapphire and silicon. These two materials have been first selected for a future cryogenic interferometer [5]. Sapphire offers a perfect transparency at 1064 nm and a very low dn/dT. Moreover, cooled sapphire payloads have been already studied in Japan as part of a proposal to construct the large-scale cryogenic gravitation wave telescope (LCGT) [6].

Silicon substrate is an alternative to sapphire at low temperature and has, by now, demonstrated very good properties in a cryogenic environment: a thermal expansion that crosses zero around 18K, low mechanical losses, and other performances comparable to the ones of the sapphire substrates. However, the silicon wavelength needs to be changed from 1064 nm to 1550 nm to get a very good transparency. It leads to a thicker coating and a most important source of noise [6]. The coating for Sapphire is a multilayer $(HL)_{17}$ HLL coating made of Ti:Ta_2O_5 (defined by H for High refractive index material) and SiO_2 (defined by L for Low refractive index material) quarter wavelength layers. It corresponds to a transmission of 6 ppm. In the case of Silicon the multilayer is a $(HL)_{19}$HLL to get the same transmission.

The mirror, studied in this note at 10K, has a diameter of 45 cm that assumes a reduced radius beam size of 9 cm. At the same time, we keep the overall test mass weight at about 200 kg. This mass leads to a silicon test mass thickness of 50 cm approximately. These dimensions have been proposed by Stefan Hild in [7] as a plausible configuration for the future Einstein Telescope. At our knowledge, formula of the thermo-refractive noise has not been studied for a finite case, thus, all the simulations in this note have been done for an infinite case.

The table 1 below considers the configuration of mirror (dimension, environment, etc).

Substrates	Silicon @ λ = 1550 nm and Sapphire @ λ = 1064 nm
Mirror's dimensions	Diameter : 45 cm, Thickness : 50 cm, Mass : ~200 kg
Shape and size of the beam	LG00, w= 9 cm
Temperature	10K
Coating	SiO_2-TiTa_2O_5 $(HL)_{19}$HLL for Silicon $(HL)_{17}$HLL for Sapphire

Table 1.Mirror configuration

The table 2 below summarizes the present knowledge of all the relevant parameter's values at 10K and at the working wavelength (see [6] for more details) of the silicon and sapphire substrate.

	Silicon	**Sapphire**
Loss angle	1 10^{-9}	4 10^{-9}
Density (kg.m-3)	2331	3997 @ 20K
Thermal conductivity (W.m-1.K-1)	2330	1500 @ 12.5 K
Specific heat (J. K-1. Kg-1)	0.276	0.0934
Thermal expansion coef. (K-1)	4.85×10^{-10}	5.3×10^{-10}
Thermo optic coef.	5.8×10^{-6} @ 30K (this value will be used for all simulations)	9×10^{-8}
Young's modulus (GPa)	162.4	464
Poisson's ratio	0.2205 @ 30K	0.23 (estimated)
Refractive index	3.45 @ 30K	1.75

Table 2. List of the values of materials substrates parameters at low temperature and good working wavelength

In the table 2, we can remark that some parameters are unknown at 10K. In this case, the temperature is specified. It is notably the case for the thermo optic coefficient. This parameter plays an important role in the thermo refractive formula, unfortunately, at our knowledge, it exists no literature with the thermo optic coefficient at 10K. Nevertheless, an extrapolation can be performed thanks to the formula:

$$n^2(\lambda,T)-1=\sum_{i=1}^{3}\frac{S_i(T)\lambda^2}{\lambda^2-\lambda_i^2(T)} \tag{3.1}$$

given in [8]. The formula is correct until 20K, thus, figure 1 shows the thermo-optic coefficient from 20K to 290K at 1.5 µm. For the sapphire, different values have been given in the literature. The Japan group [9] found a very low thermo-refractive coefficient of 9 10^{-8} 1/K. This value has been used to do calculation in this note. Other literature [10] gives also some point for the dn/dT of the sapphire substrate. Thanks to an interpolation of these 6 points it is possible to get the data from 25K to 300K. At 25K the dn/dT of the sapphire is around 10^{-7}. It confirms the value mentioned by [9].

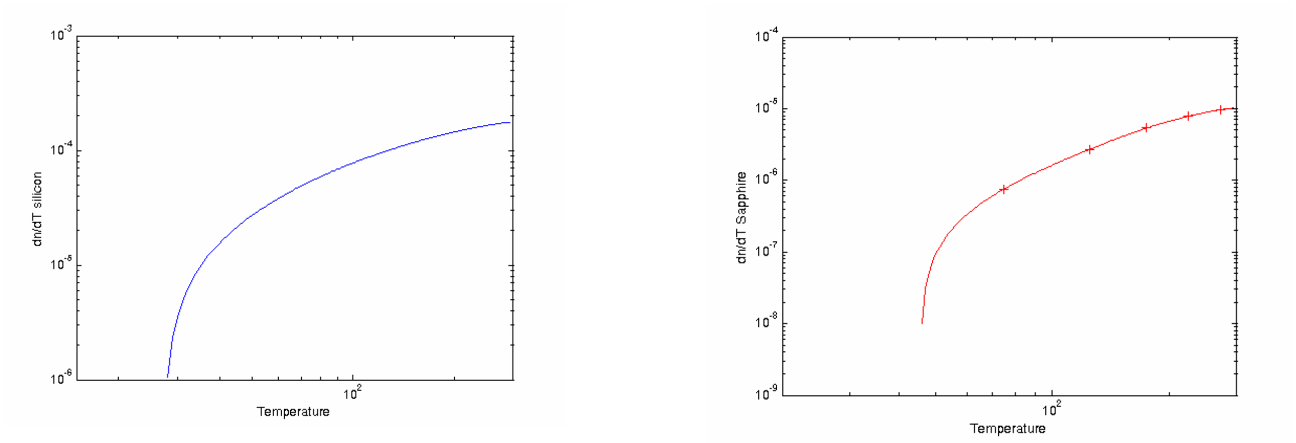

Figure 1. Thermo-optic coefficient of Silicon (a) [8] @ 1.5 µm and Sapphire (b) [10] @ 1.064 µm from 30 to 290K

4. Simulations

For this section, we propose to take into account the realistic data of the different parameters taking place in formulas and to check the evolution of the total thermal noise vs temperature. Table 3 lists the parameter values depending on temperature.

	Coating loss angle	Thermal conductivity (W.m-1.K-1)	Specific heat (J. K-1. Kg-1)	Thermal expansion coef. (K-1)
10K	Ti:Ta2O5 : 3.8 10^{-4} & SiO2 : 5 10^{-4}	2330	0.276	4.85 10^{-10}
20K	Ti:Ta2O5 : 8.8 10^{-4} & SiO2 : 5 10^{-4}	5022.3	3.3665	-2.89 10^{-9}
30K	Ti:Ta2O5 : 4.2 10^{-4} & SiO2 : 5 10^{-4}	4821.3	17.017	-5.28 10^{-8}
40K	Ti:Ta2O5 : 4 10^{-4} & SiO2 : 3 10^{-4}	3536.6	43.924	-1.64 10^{-7}

Table 3. Values of the silicon substrate parameters having variation with temperature

Values for thermal conductivity, specific heat, thermal expansion coefficient come from the material properties database (JAHM software). About coating loss angles, they have been estimated from the graphs of the figure 2. Figure 2 shows the evolution of the mechanical losses angles of the two coating materials: $TiTa_2O_5$ and SiO_2 studied from 0K to 300K [11]. For $TiTa_2O_5$, we can observe that loss angles attain a peak around 20K and then, decreases from 20K to 300K. For SiO_2 layer, the value is stable from 5 to 30 K and then, tends to decreases until to reach 5 10^{-5} at 300K.

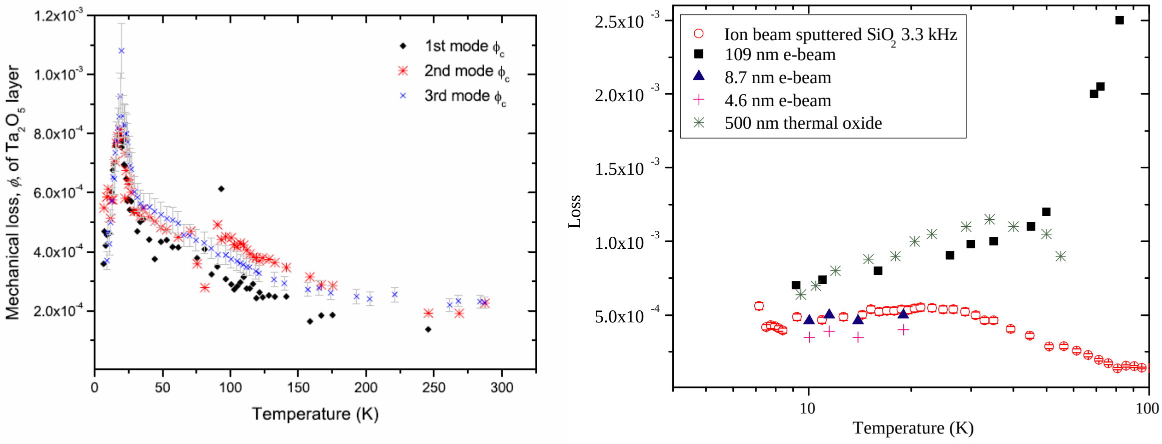

Figure 2. Temperature dependence of the measured mechanical loss of $TiTa_2O_5$ and SiO_2 thin films

Figure 3 shows the evolution of total thermal noises on silicon and sapphire with and without the adiabatic assumption according to the formula of φ_n 1.1 and 1.2 respectively, explained in section I of this note. The dashed lines use the standard equation referred by [2] for the interferometer, it corresponds to the adiabatic approximation. The continuous lines, on contrary, refer to the non-adiabatic case. Without adiabatic assumption the thermo-refractive noise tends to decrease at low frequencies. At higher frequency, the two spectral densities are the same. The results in figure 2 show that, from the point of view of substrate thermo-refractive noise, sapphire is clearly better at cryogenic temperatures. Nevertheless, at low temperature, Substrate Thermo Refractive noise of Silicon test mass is also below the ET sensitivity target. The ET sensitivity calculations have been explained in [7].

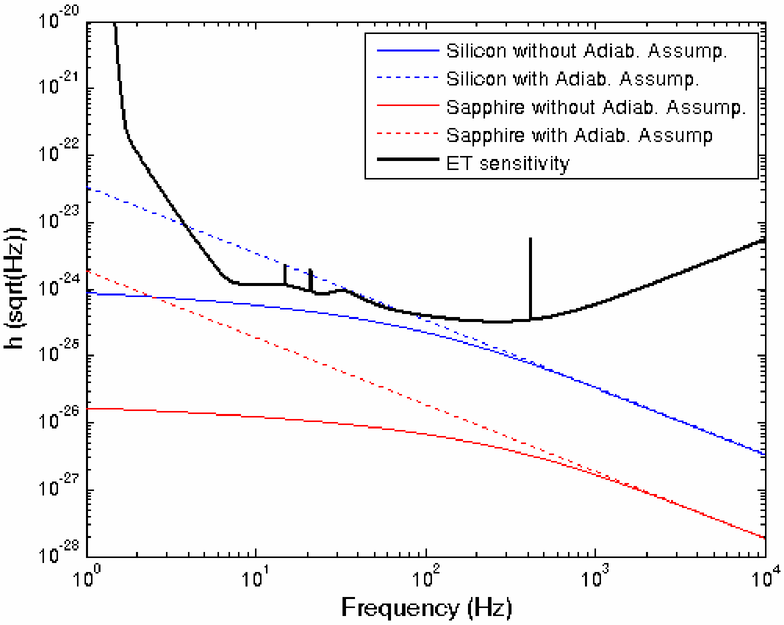

Figure 3. Evaluation of substrate thermo-refractive noise with and without adiabatic assumption

Figure 4 shows the evolution of substrate thermo-refractive for Silicon test mass at different temperatures 10K, 20K, 30K and 40K. The thermo-optic coefficient has not been changed (5.8 10^{-6} 1/K). Only the specific heat and the thermal conductivity, which are two parameters that exhibit important variations at low temperatures, can modify the formula. Note that the substrate thermo refractive noise decreases when the temperature increases.

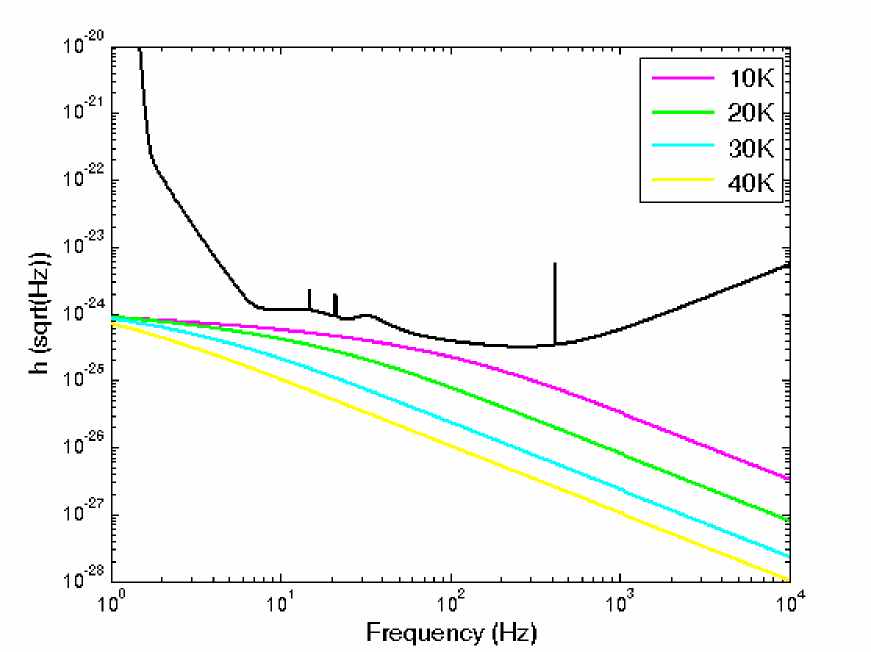

Figure 4. Silicon substrate thermo refractive noise at different cryogenic temperatures: 10K, 20K, 30K, 40K

However, if the silicon thermo-refractive noise tends to decrease when the temperature increases, it is not the case for all the thermal noises considered (Brownian and Thermo elastic Noise for coating and substrate). Other parameters values vary with temperature as loss angles, thermal expansion coefficients, specific heat, etc as seen in the table 3 of this note. These modifications are not necessarily favourable for the final total thermal noise.

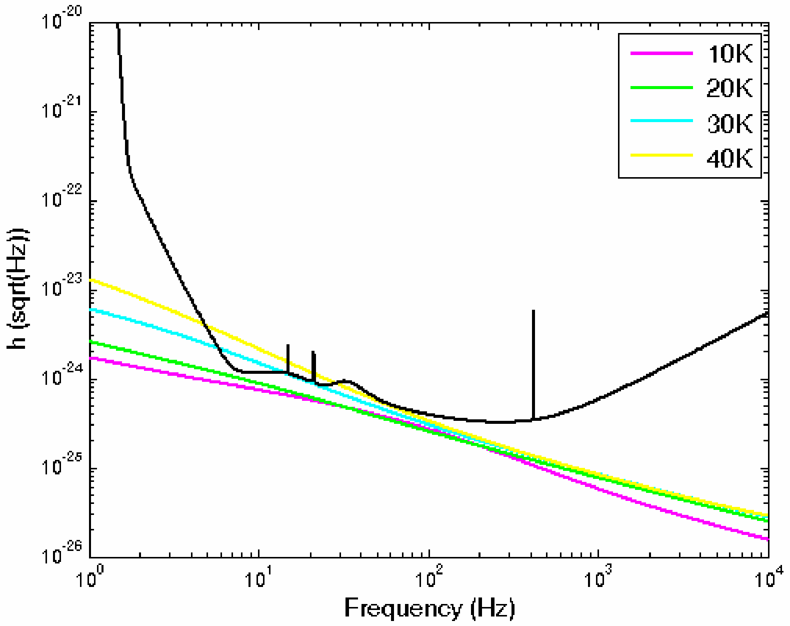

Figure 5. Total thermal noise at different cryogenic temperature for an interferometer using silicon substrates

We remark in figure 5 that for 30K and 40K, the total thermal noise is above the ET sensitivity target from 5 to 30 Hz approximately. The different thermal noises considered can limit the total thermal noise because of the parameters values changing with temperature. For example, at 10K and 20K, the substrate thermo refractive noise and the Coating Brownian noise limit the total thermal noise. At 30K and 40K, the noise is limited by the Substrate thermo elastic and Coating Brownian noise also. The contribution of the different noises at different temperatures is given in the appendices.

5. Conclusion

Silicon and Sapphire substrates are two promising solutions proposed for future cryogenic interferometer based on Fabry-Perot cavities. The note was devoted to the estimation of Thermo refractive noise for future ET mirrors. We can conclude that Thermo-refractive noise of Silicon substrate is higher than the one of Sapphire principally due to the thermo-optic coefficient value. Although the both noises are below the ET sensitivity target, it is important to mention that dn/dT used in this note were; 9 10^{-8} 1/K for Sapphire and 5.8 10^{-6} 1/K for Silicon. A doubt can be argued concerning the reality of the value of dn/dT of Silicon at 10K because this latter was used known only at 30K. The value measured at 30K has been used for all simulation of this note. Therefore, we can expect that the thermo refractive noise of silicon presented here decreases with future dn/dT measurement at 10K. These measurements should be performed in the next months.

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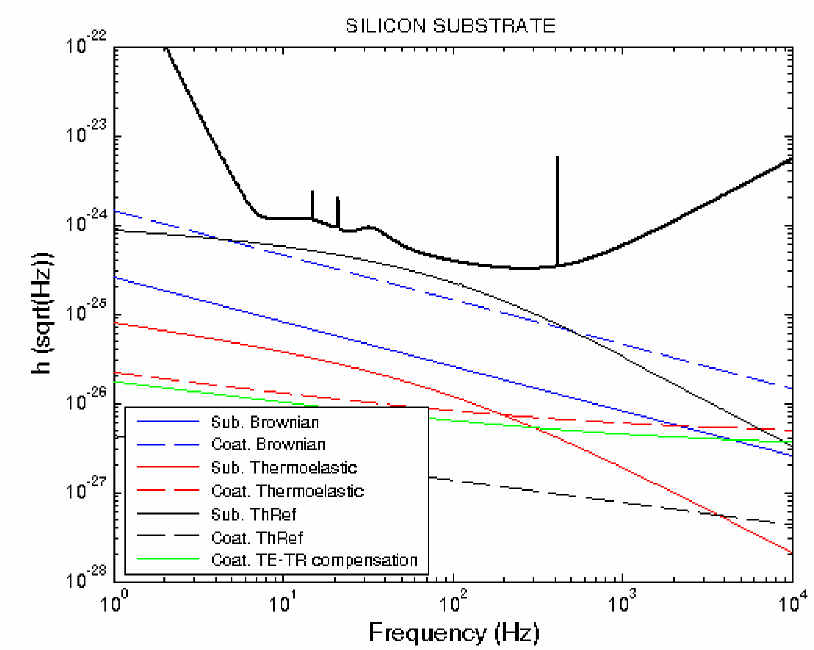
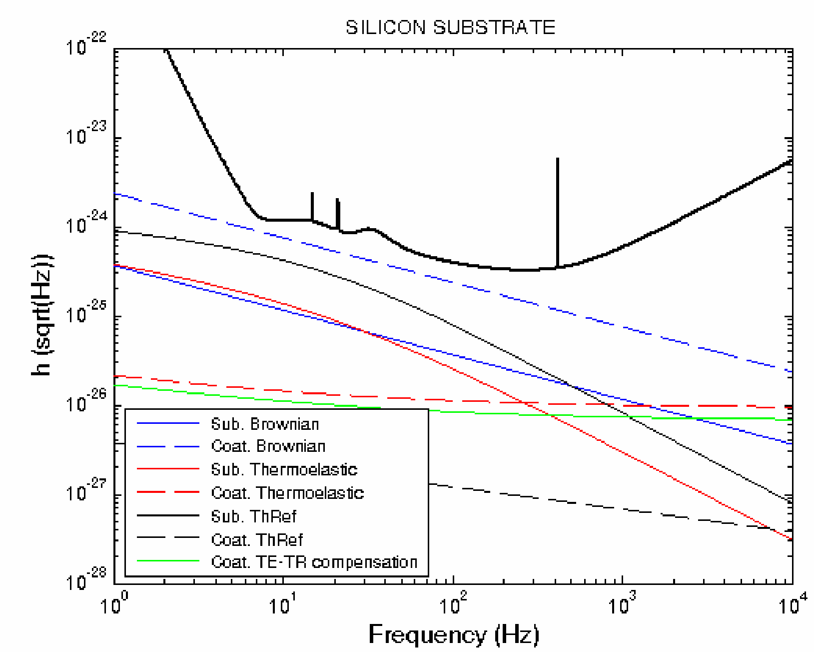
APPENDICES



Thermal noises contribution at 10K – Comparison with ET sensitivity target



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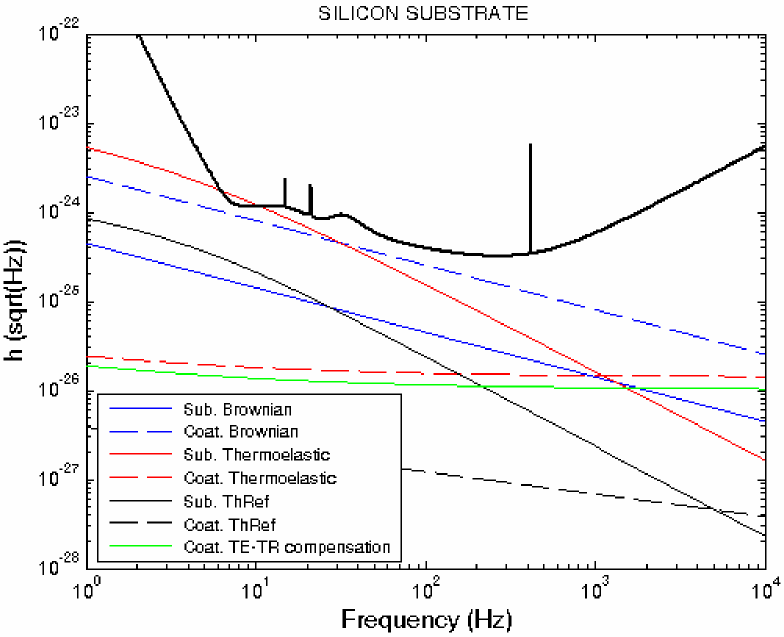

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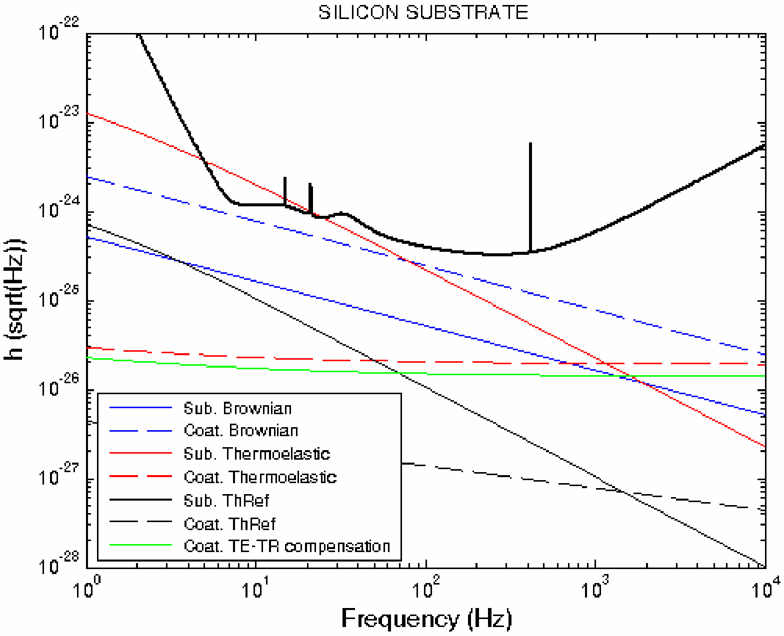

Thermal noises contribution at 40K – Comparison with ET sensitivity target