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Analysis of the resonant modes present in a novel cryogenic test-mass suspension for the Einstein Telescope Gravitational Wave Observatory

Technical Report

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Abstract

This technical report expands on a broader analysis of a novel cryogenic suspension designed for third-generation gravitational-wave detectors. Specifically, it investigates the effects of geometric modifications to two key components that this system incorporates: a suspension beam and flexures operating under tension and compression, respectively. Using Finite Element Method (FEM) simulations in COMSOL and ANSYS, this study examines how variations in beam thickness and flexure aspect ratios influence the system's natural resonant modes and some of its mechanical behavior.

By systematically exploring these parameters, we identify configurations that minimize undesirable mechanical conditions and provide a basis for future studies, such as the flexure design from group 1 discussed in Section 3. Additionally, we analyze how beam thickness variations influence vibrational attenuation, aiming to extend the effective attenuation range by increasing the frequency of the second resonant mode, discussed in Section 4. Finally, we identify regions of high resonance amplitudes as potential sites for implementing active damping and cooling strategies, discussed in Section 5.

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1 Introduction

This technical report is a companion to a paper entitled "A Cryogenic Test-Mass Suspension with Flexures Operating in Compression for Third-Generation Gravitational-Wave Detectors" [1]. The paper provides the first conceptual analysis of a novel suspension design for the cryogenic test-mass mirrors of the Einstein Telescope gravitational-wave observatory. This design features vertical rigid beams and flexures that operate under tension and compression, respectively. This study focuses on modifications to these components and examines the resulting resonant modes and their characteristics. An illustration of the design can be found in Figure 1, which outlines some of the different components.

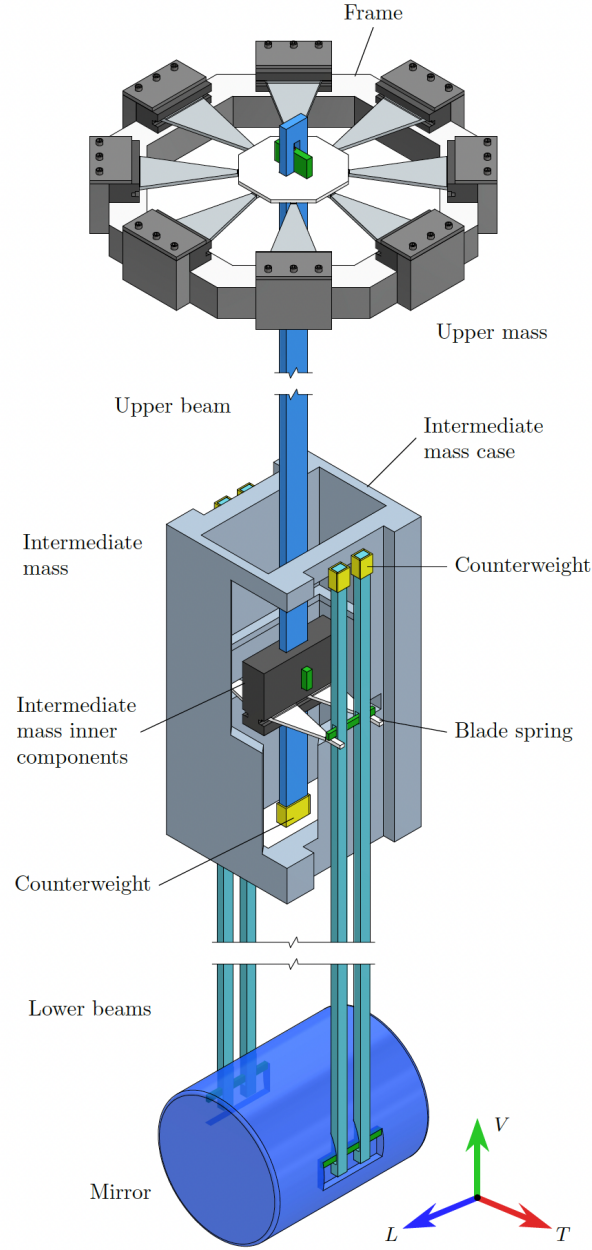


Figure 1: Illustration of the proposed suspension system, highlighting its components. The coordinate system defines the main directions as follows: L for longitudinal, V for vertical, and T for transverse.

This report specifically examines the natural resonant modes of the proposed system, analyzing how changes in beam thickness and flexure aspect ratios influence these modes and impact the mechanical behavior, providing further insight into the reasoning behind some of the statements made in the paper. To predict these results, the Finite Element Method (FEM) was employed using COMSOL and ANSYS software for cross-validation of the findings.

Furthermore, this study highlights the significant impact of beam thickness vari-

ations on the system’s mechanical response. As increasing the range in which vibrational attenuation happens is highly desirable, Section 4 explores how an increase in this range can be achieved by increasing the thickness of the beam in the longitudinal direction, as there exists a linear dependency between the thickness of the beam and the frequency of the normal mode that disrupts the effective attenuation range.

Finally, Section 5 describes an investigation of the normal mode shapes for identifying the places of the beam where the displacement is higher, revealing suitable locations for placing the components of an active control system for damping and cooling the modes.

2 Beam and flexure system

The simplified pendulum we analyze comprises the beam and the two flexures shown in Figure 2, which corresponds to the lower beam in Figure 1. This is a rigid beam working in tension, with windows hosting two flexures working in compression.

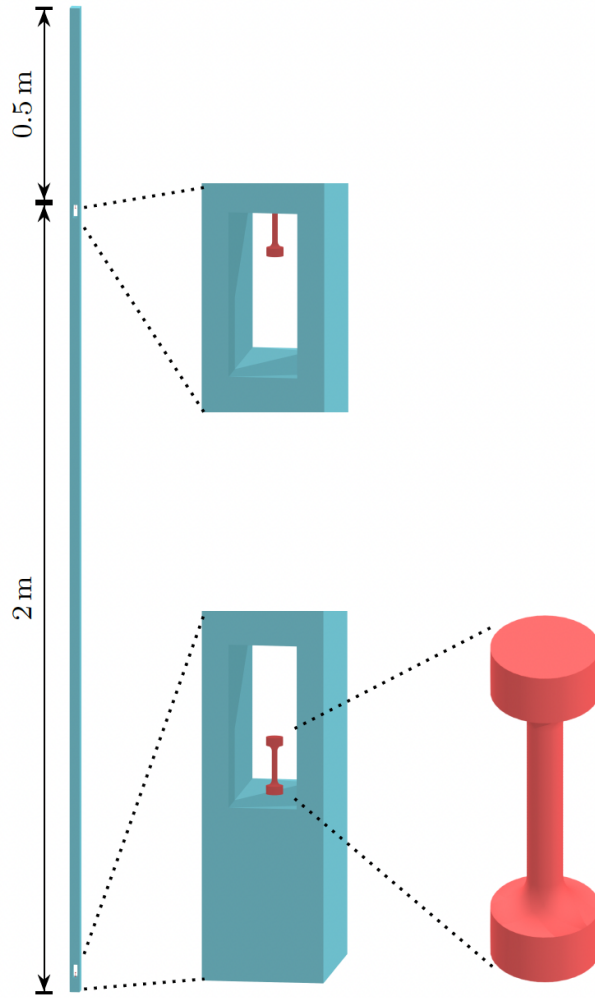


Figure 2: Simplified beam and two flexure system.

The boundary conditions set on the FEM software programs for our simulations are as follows:

- A counterweight of 0.65 kg is placed at the top face of the beam (except for the calculations from Section 4, where the counterweight value is 0.86 kg.)
- A movement constraint in all 6 degrees of freedom on the bottom face of the top flexure, from which the beam is suspended. Meaning that the face is completely fixed in place.
- A movement constraint in the three angular degrees of freedom on the top face of the bottom flexure, to account for the angular rigidity due to the mirror being suspended from the four beams, as depicted in Figure 1.

- A load of 50 kg is placed at the top face of the bottom flexure to simulate the contribution that a 200 kg mirror has on each beam.
- Acceleration due to gravity in the vertical direction using the coordinate system displayed Figure 1.

The calculations also considered the anisotropic stiffness matrix of single-crystal silicon [2], assuming a bulk loss angle of $\phi = 10^{-9}$. Other loss contributions were not considered. The crystal orientation with Miller index [100] was aligned vertically for all components using the coordinate system in Figure 1.

Under these conditions, ANSYS and COMSOL predict the frequency values and shapes of the normal modes.

The normal modes are determined by solving the eigenvalue problem that arises from the system's equations of motion for free vibrations, as outlined in [3], while considering the previously defined boundary conditions. For our simplified system, the equation of motion is expressed as:

$$M\ddot{x} + Kx = F(t), \quad (1)$$

where M represents the mass matrix, K denotes the stiffness matrix, x is the displacement vector, $\ddot{x}$ is the acceleration vector, and $F(t)$ is the time-dependent external force vector. In the case of free vibration, the external force is absent, leading to $F(t) = 0$.

ANSYS and COMSOL (used in parallel as a means of cross-checking the results) address the undamped eigenvalue problem to compute the natural frequencies and corresponding mode shapes, starting from:

$$M\ddot{x} + Kx = 0. \quad (2)$$

To solve this, a harmonic solution is assumed in the form:

$$x = \Phi e^{i\omega t}, \quad (3)$$

where Φ is the mode shape (represented by the eigenvector) and ω is the natural frequency. Substituting this into the Equation 2, ANSYS and COMSOL evaluate the following:

$$(K - \omega^2 M)\Phi = 0, \quad (4)$$

to determine the system's natural frequencies and mode shapes. After the modal

analysis, the displacement transfer functions in the longitudinal direction were computed using both programs.

To obtain the transfer functions, a pre-stressed harmonic analysis was performed in the frequency domain, where the bottom face of the top flexure was subjected to a harmonic acceleration a_0 , and the resulting acceleration a at the top face of the bottom flexure was obtained. The displacement transfer function T was then derived from this acceleration transfer function:

$$T = \frac{a}{a_0} = \frac{-\omega^2 s}{-\omega^2 s_0} = \frac{s}{s_0}, \quad (5)$$

where ω is the angular frequency, s_0 and s represent the input and output displacements, respectively. Lastly, we only present results for the longitudinal direction, which is the interferometer’s beam direction, which is also defined in Figure 1. Resonances in other directions are disregarded.

3 Frequency Analysis for different flexure dimensions

As described with more detail in another accompanying technical report [4], the definitions of the key parameters of the analyzed cases are listed in Table 1.

Case name	Safety factor (nominal)	Element diameter (mm)	Element length (mm)
1-1	3.1	0.83	1.7
1-2	3.1	0.83	3.4
1-3	3.1	0.83	5
2-1	6.1	1.16	6
2-2	6.1	1.16	9
2-3	6.1	1.16	12
3-1	18.1	2.0	10
3-2	18.1	2.0	24
3-3	18.1	2.0	38

Table 1: Flexure parameters for the study.

This section explores the normal modes of each case and their effect on the displacement transfer function. A notable characteristic common to all cases is the pendulum mode observed at frequencies of 356 mHz. This resonant mode is determined by the length of the pendulum and remains largely insensitive to variations in flexure stiffness. High-frequency modes begin to exhibit characteristics related

to the flexure's stiffness. As we will see in Figure 4(a), the first of these modes is particularly important because it disrupts the $1/f^2$ dependency in the displacement transfer function, which follows the initial resonance caused by the pendulum mode. The range between the first and second resonances, where the $1/f^2$ dependency is present, is where effective attenuation occurs. Therefore, exploring these modes is of significant importance.

Describing the shapes for high-frequency modal shapes becomes necessary. Figure 3 illustrates examples of the shapes that are encountered in the sections to follow and their distinct characteristics.

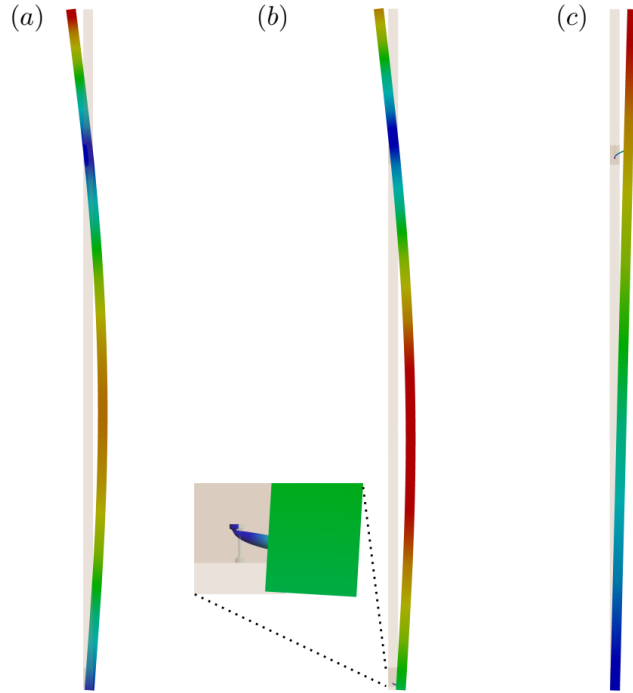


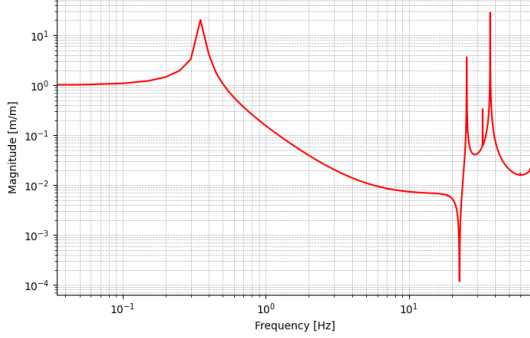
Figure 3: Panel (a) displays the banana mode, characterized by two null points in the beam and flexure. Panel (b) displays the hybrid mode, that also presents two null points but only one of them is present at the beam. Panel (c) shows a rigid body mode, in which only the upper flexure is deformed. The color scale indicates deformation, in which red corresponds to the largest.

Based on the distinct characteristics of the system's normal modes and the shapes of their displacement transfer functions, we have categorized our cases into three distinct groups.

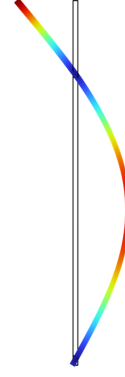
3.1 Group 1: The Banana Mode at around 25 Hz

The cases from this group feature a pendulum mode at 356 mHz, followed by a banana mode at approximately 25 Hz. The cases in which this behavior is observed

are **1-1, 1-2, 2-1 and 3-1**. As an example, Figure 4(a) shows the transfer function for case 2-1, where we note the resonance at 356 mHz of the pendulum mode, followed by the peak of the banana mode present at around 25 Hz. As shown in Figure 4(b), this mode features two null points in the beam and flexures.



(a) Displacement transfer function of case 2-1.



(b) Shape of the banana mode for case 2-1. It is also shown in Figure 3(a), but the deformation is exaggerated here to a larger extent.

Figure 4: For the case 2-1, representative of group 1, panel (a) shows the displacement transfer function, and panel (b) shows the shape of the first resonant mode that breaks the $1/f^2$ dependency, the banana mode. These were simulated using COMSOL.

To quantify this mode across the group, the banana mode frequencies for each case are calculated. These values, along with the average frequency and maximum relative deviation, are summarized in Table 2.

Case	Frequency of the banana mode(Hz)
1-1	25.97
1-2	25.22
2-1	25.23
3-1	25.77
Average frequency	25.55
Maximum relative deviation (%)	1.659

Table 2: Frequency values and statistics for the banana mode present in the cases from group 1.

The maximum relative deviation was computed to assess how much the frequencies differ from each other and to measure the worst-case variability within the

group, highlighting the extent to which the most extreme frequency departs from the average behavior.

For example, for this group, the mean frequency F_{mean} is determined to be

$$F_{\text{mean}} = 25.55 \text{ Hz} \quad (6)$$

The frequency furthest from the mean, denoted F_{far} , is identified as the value from case 1-1:

$$F_{\text{far}} = F_{\text{case 1-1}} = 25.97 \text{ Hz}. \quad (7)$$

The maximum relative deviation percentage is then calculated to be

$$\text{Maximum relative deviation} = 1.659\%. \quad (8)$$

A small value indicating tight clustering of the banana mode frequencies in group 1. This suggests that for this particular group, changes in the flexure's aspect ratio produce a small effect on the frequency value of the banana mode.

3.2 Group 2: Presence of Intermediate Modes—Hybrid Modes

Compared with the first group, the cases in this second group exhibit additional resonances within our range of interest between the frequencies of the pendulum and banana modes of group 1. The shapes of these resonances resemble the hybrid mode shown in Figure 3(b) where a significant deformation is present not only in the beam but also at the flexures. The cases in this group are **1-3**, **2-2**, **3-2**.

The frequency values for the resonances present in this group can be found in Table 3, along with their averages and the maximum relative deviations.

Case	Frequencies of the hybrid modes (Hz)	
	1st Mode	2nd Mode
1-3	19.8	46.94
2-2	22.67	55.63
3-2	21.06	49.90
Average frequency	21.177	50.823
Maximum relative deviation (%)	6.501	9.458

Table 3: Frequencies and statistics for the hybrid modes present in cases from group 2.

The transfer function for case 3-2, used as a representative case out of this group,

is seen in Figure 5. The shapes and frequency values for these normal modes can be found in Figure 6.

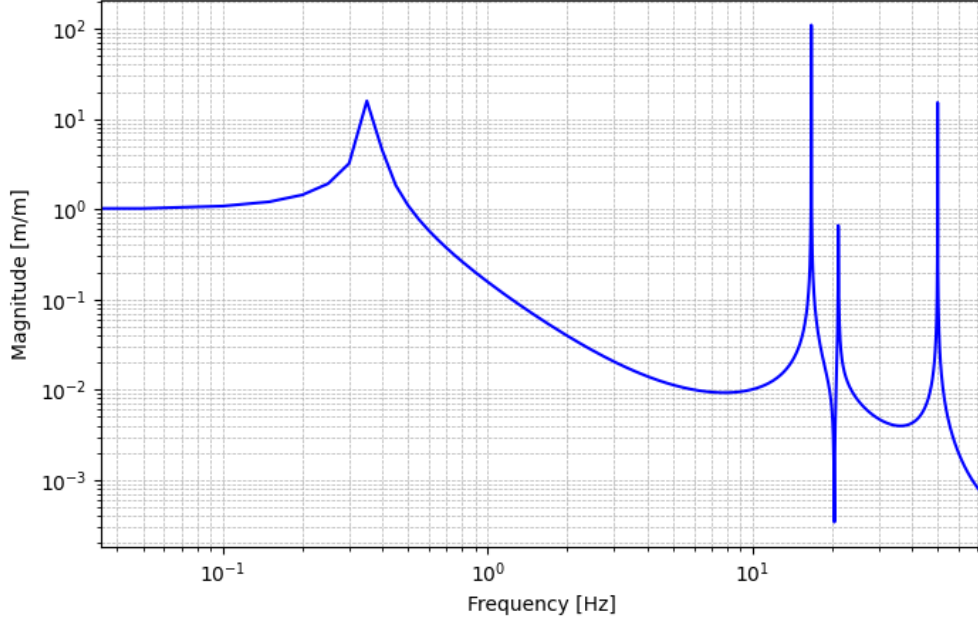
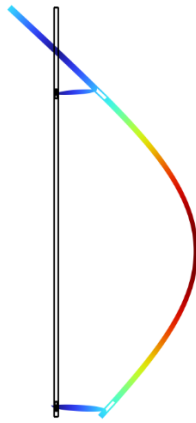
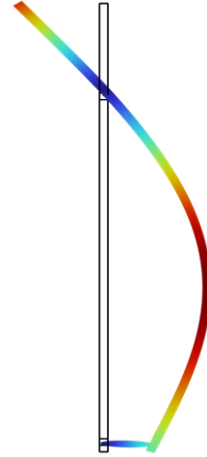


Figure 5: Displacement transfer function of case 3-2.



(a) Hybrid mode for case 3-2 at 17.15 Hz.



(b) Hybrid mode for case 3-2 at 21.05 Hz.

Figure 6: Shapes of the resonant modes present in the displacement transfer function of case 3-2.

By visual inspection of Figure 6, one can note that the modes display characteristics similar to those of the banana mode, like a highly deformed beam. The

difference between the modes from this group is the amplitude of deformation that the flexures present. The one in Figure 6(a) presents large deformations in both upper and lower flexures, whilst in the one in Figure 6(b), only the lower flexure experiences a large deformation.

The maximum relative deviations for this group are 6.5% for the lower frequency hybrid mode, and 9.5% for the higher frequency hybrid mode, indicating a greater variability compared to group 1. These larger deviations reflect the influence of increased flexure deformation, which introduces more variation in the resonant frequencies across the cases.

3.3 Group 3: Significant Low-Frequency Rigid-Body Modes.

In the third group of cases, we see the introduction of a pure rigid-body mode, as illustrated in Figure 3(c), followed by two other resonant modes present in the range of interest. The observed shape of these other resonances has characteristics similar to the hybrid mode seen in group 2 but with additional contributions that resemble a rigid-body mode. We've named these hybrids as well. The cases in this group are **2-3 and 3-3**.

For example, the displacement transfer function for case 3-3, displayed in Figure 7, in addition to a low-frequency rigid body mode present at 4 Hz, reveals two peaks that correspond to two hybrid modes. The shapes of the initial rigid body mode and hybrid modes that follow can be found in Figure 8.

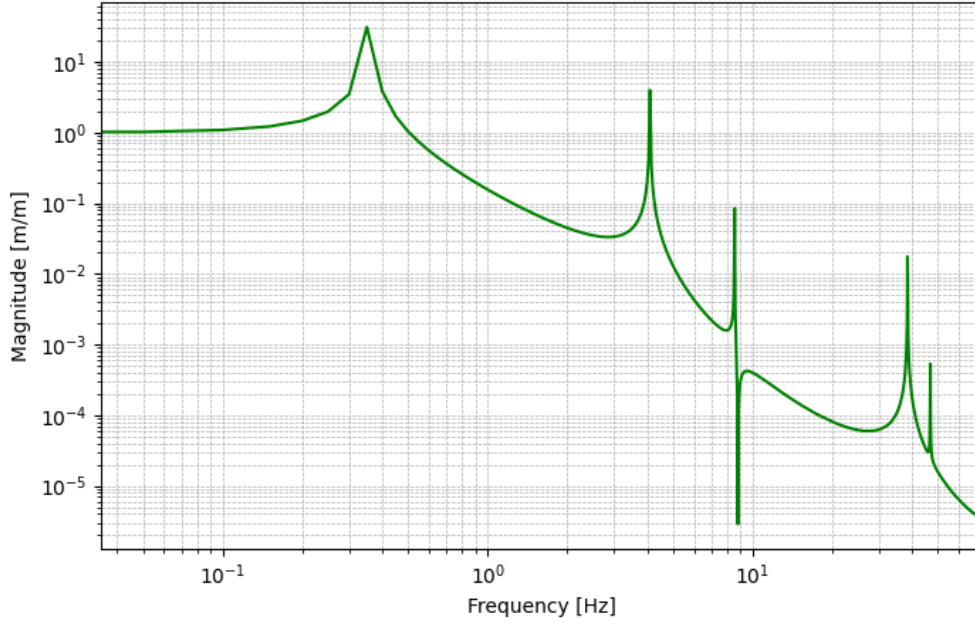


Figure 7: Displacement transfer function of case 3-3.

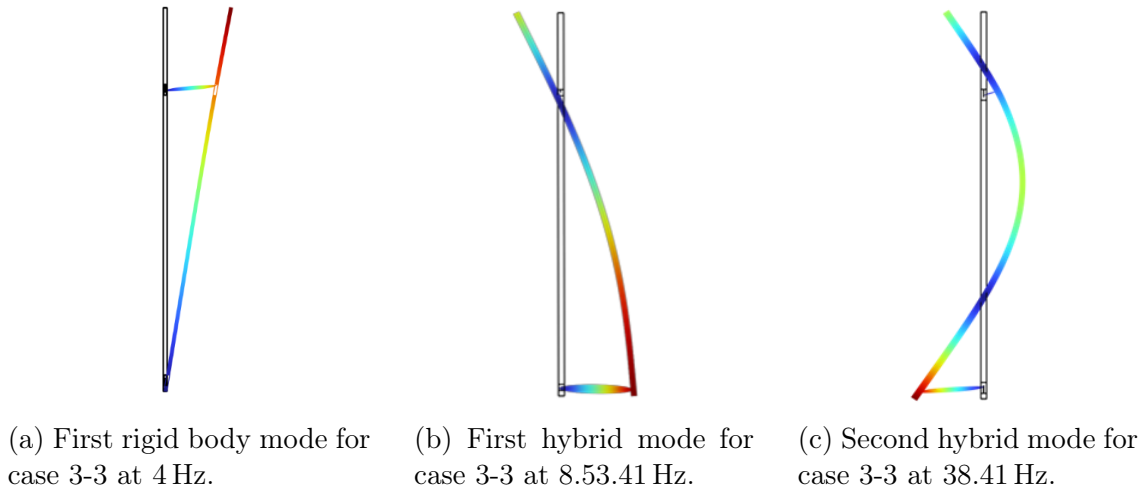


Figure 8: Shapes of the three resonant modes present in the displacement transfer function of case 3-3.

By visual inspection of Figure 8, one can note that the mode displayed in Figure 8(a) resembles the rigid-body modes in Figure 3(c), whilst the hybrid mode that follows presents characteristics that resemble the banana mode, but with very high deformations present in both flexures. This high deformation comes from the extended length and therefore, lack of stiffness of the flexures in this group present.

The frequency for these modes for both cases can be found in Table 4, along with their averages and maximum relative deviations.

Case	Frequencies of the hybrid modes (Hz)	
	1st mode	2nd mode
2-3	14.84	41.08
3-3	8.53	38.41
Average frequency	11.685	39.745
Maximum relative deviation (%)	27	3.359

Table 4: Frequencies and statistics for group 3.

For the first hybrid mode, the maximum relative deviation is 27%, derived from its mean of 11.685 Hz and the furthest value of 14.84 Hz from case 2-3. The second hybrid mode presents less variability. Besides the presence of undesirable resonant frequencies in the range of interest, rigid-body modes at lower frequencies have been seen identified to conditions in which the pendulum mode buckles, making them highly undesirable. More details are available on another accompanying technical report [4].

4 Frequency analysis for different beam dimensions.

Another parameter to consider is the behavior that the different normal modes can have when changing the geometry of the beam; we are particularly interested in the longitudinal direction.

To investigate this, we first consider a beam with a width of 25 mm in the transverse direction and an initial thickness of 20 mm in the longitudinal direction, with two flexures of 1.081 mm in diameter and lengths of 9 mm.

Additionally, for this study alone, the counterweight used for the system is 0.86 kg for all thicknesses. We make 5 mm increments to the thickness of the beam in the longitudinal direction until a thickness of 35 mm is reached. This is illustrated in Figure 9. These results can be found in Table 5.



Figure 9: Width (black) and thickness (red) of the suspension beam. Where black and red represent the transverse and longitudinal direction, respectively. The increased was performed in the longitudinal direction (red).

Shape	Frequency values (Hz) for each beam thickness			
	20 mm	25 mm	30 mm	35 mm
Pendulum mode	0.3578	0.358	0.3583	0.3585
Banana mode	12.879	16.577	20.235	23.73
Hybrid mode	25.19	28.6	30.26	30.66
S-mode	64.23	69.13	72.47	74.75

Table 5: Frequencies of longitudinal modes across different thicknesses.

A plot of the frequency values of the normal modes across the beam thicknesses can be found in Figure 10, where a linear trend can be noted.

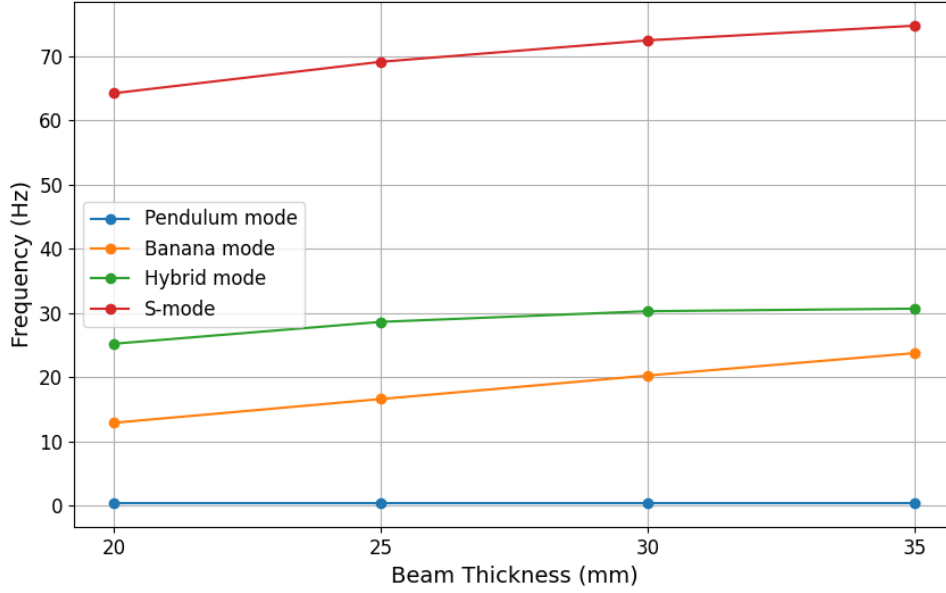


Figure 10: Frequency values across normal modes for different beam thicknesses.

To quantify this trend across all modes, the coefficient of determination R^2 is computed for all modes, providing a measure of how well the linear fit explains the relationship between the thickness of the beam and the frequency of the different modes. A R^2 close to 1 means the frequency increases linearly with thickness, while a lower value suggests a weaker linear trend. These results can be found in Table 6.

Mode Shape	R^2 Value
Pendulum mode	0.9895
Banana mode	0.9992
Hybrid mode	0.9088
S-mode	0.9875

Table 6: R^2 values show how well frequency scales linearly with beam thickness. A value near 1 indicates a strong linear fit.

The increase in the frequency of the natural modes has an effect on the mechanical behavior of our system. To help illustrate this, we present in Figure 11 the displacement transfer functions for our increasing beam thickness cases. We note that the frequency variation of the first resonance, which is the pendulum mode, is negligible, whilst the second resonance, as the thickness is increased, moves to higher frequencies. This resonant frequency is of special importance as it disrupts the $1/f^2$ dependency that starts after the first resonance, thus determining the range

in which effective vibrational attenuation occurs. Maximizing this range is highly desirable. Therefore, a thickness of at least 35 mm is chosen for future study.

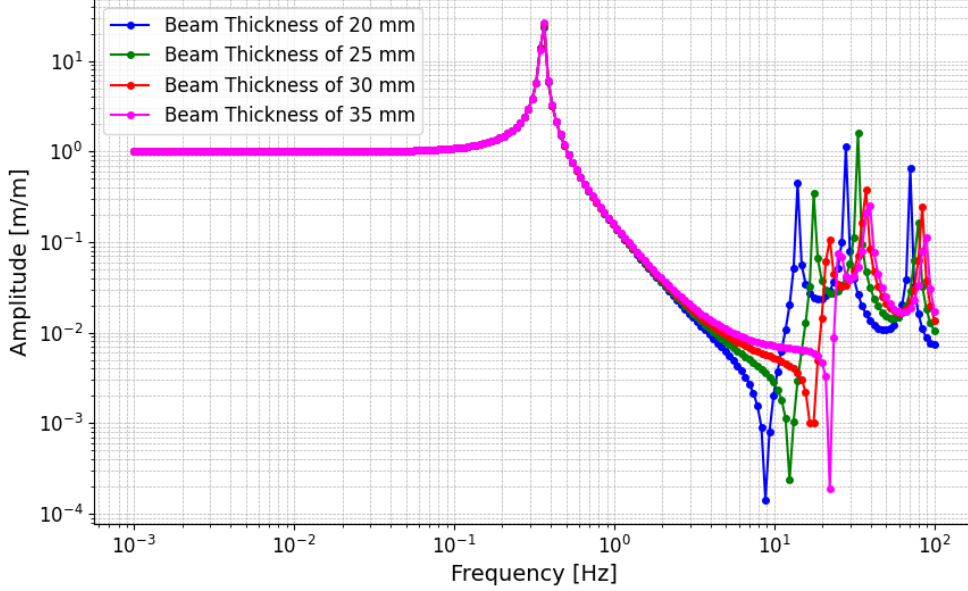


Figure 11: Displacement transfer functions of the different cases we considered in this section, in which the thickness of the longitudinal component was increased to 35 mm.

5 Relative amplitude of the different resonant modes

In the context of finding convenient locations to place actuators that provide active damping and cooling, a factor to consider is the amplitude of the different resonances.

Considering the beam and flexure system described in Figure 2, and the flexure case 2-1 from Table 1, we select three points along the pendulum in which we will probe their relative displacement for different modes:

- Upper face of the bottom flexure (where the test mass is placed).
- Upper face of the beam.
- 0.5 m below the alcove that hosts the upper flexure.

It is important to note that the middle point was chosen based on a physical boundary, where the hypothetical actuator would extend only as far as the intermediate mass, approximately 0.5 m below the upper alcove of the beam, as illustrated in Figure 1. These points are highlighted in Figure 12.

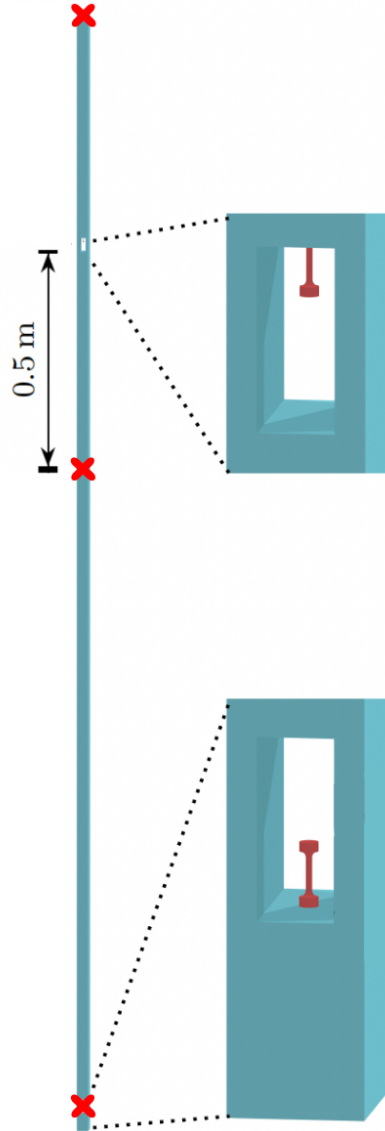


Figure 12: Locations in which the relative displacement probes for this amplitude calculation are highlighted in red Xs. The first point evaluates the motion imposed by the resonance on the mirror. The second and third points correspond to the upper and lower end of the intermediate mass, on which the sensor and actuator can be mounted to.

In obtaining the amplitudes of the modes at the different points, it is important to note that we are performing a modal analysis formulated as the eigenvalue problem described in Section 2. According to Equation (4), the calculation returns the eigenvalue ω^2 , where ω is the natural angular frequency, and the shape of the mode as the eigenvector Φ . This equation does not determine the amplitude of the eigenvector, as any eigenvector multiplied by a scalar is also an eigenvector. FEM software, COMSOL used in this instance, assigns an arbitrary reference scaling to the eigenvectors to present displacement values. These values are not absolute but

serve as a way to visualize mode shapes. Therefore, the displacements should be interpreted in a relative sense only, in which only the displacement ratios at different points are meaningful.

As discussed, our focus is on resonances along the longitudinal direction, with vertical and rotational resonances neglected. With the simulated resonance shapes, we probe the displacement of the defined points, and the results are presented in Table 7.

Normal mode		Relative Displacement ($\times 10^{-6}$ m)		
Mode Shape	Frequency (Hz)	0.5 m below Mirror	Mirror	Top of beam
Pendulum	0.358	0.663	2.45	0.633
Banana	25.22	1.64	0.07	2.50
Hybrid	36.01	1.26	0.026	2.50
S-mode	81.86	1.93	0.046	0.768

Table 7: Relative displacement data for different modes, all simulated using COMSOL.

The shapes of the higher frequency modes can be found in Figure 13.

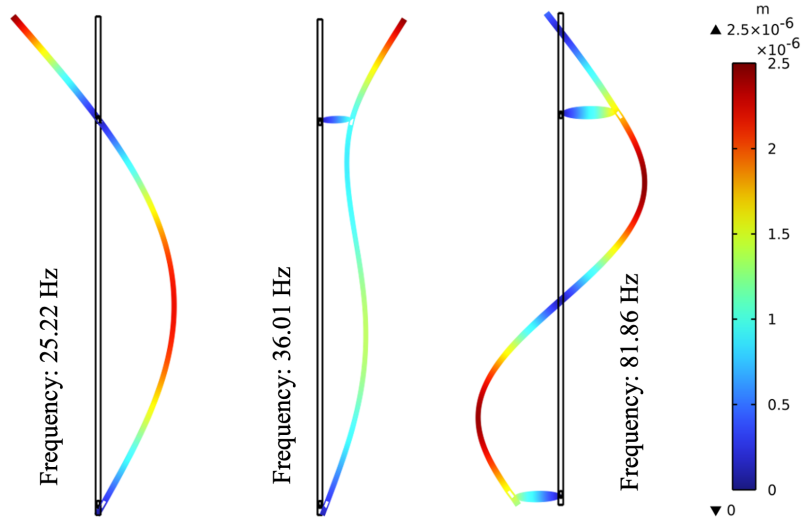


Figure 13: Shapes of the different resonances we considered. From left to right, the modes are: banana, hybrid and S modes. The scale is arbitrary and represents relative deformation, with red being the maximum deformation point. The image/scale is taken from COMSOL.

The displacement data for the various normal modes, as presented in Table 7, allows us to calculate two key ratios:

$$\text{Ratio}_{\text{top}} = \frac{\text{Displacement at top of the beam}}{\text{Displacement at mirror}}$$

and

$$\text{Ratio}_{0.5} = \frac{\text{Displacement 0.5 m below the upper alcove}}{\text{Displacement at mirror}}$$

These ratios help quantify the magnitude differences in displacement across various points in the beam for different resonances, which aid in the selection of the most effective location for active damping or cooling.

Results for these ratios between the different points along the beam can be found in Table 8.

Mode shape	Ratio _{top}	Ratio _{0.5}
Pendulum	0.2584	0.2706
Banana	35.71	23.43
Hybrid	98.04	49.41
S-mode	16.59	41.68

Table 8: Calculated displacement ratios for different modes.

The results in Table 8 and a visual inspection of Figure 13 provide useful insights into potential actuator placement for damping internal resonances. For both the banana and hybrid modes, the displacement amplitude at the top of the beam is at least 35 and 98 times larger than at the mirror, respectively. In the case of the S-mode, the highest displacement occurs near the probe positioned 0.5 m below the upper flexure, and it is 41 times larger there than at the mirror level. Given that the banana and hybrid modes are more relevant in our system, the top of the beam is the best location for an actuator to provide active damping or cooling.

6 Conclusion

This technical report provides a detailed analysis of various factors that influence some of the mechanical behavior of the proposed suspension system, with a particular focus on modifications to the flexure’s aspect ratio and beam thickness. By systematically investigating these parameters, we aim to avoid undesirable mechanical conditions, whilst identifying optimal configurations that constitute a basis for future investigation. For instance, in Section 3 we identify flexure group 1 as a very good set of cases on which to continue future studies, characterized by a pendulum mode present at 356 mHz followed by a banana mode that stays roughly at 25 Hz for all cases in the group. The second group presents additional resonances in the range of interest, with added variability. In the third group, there are resonant modes at even lower frequencies, a condition that has been found to set this group

close to buckling conditions [4]. Additionally, this technical report highlights the significant impact of variations in the beam’s thickness on the system’s behavior. Specifically, the vibrational attenuation that the system can provide. To extend the frequency range over which effective attenuation occurs, we seek an ideal case in which the first pendulum resonance occurs at very low frequencies while simultaneously shifting the second resonant mode to higher frequencies. A frequency increase in the second resonant mode can be achieved by increasing the beam thickness, as discussed in Section 4. A thickness of at least 35 mm is chosen for future study because it provides the best attenuation factor. Longitudinal thicknesses exceeding 50 mm would not only enhance attenuation performance but also shift the resonance above the ET-LF sensitivity range—a desirable outcome that may be technically challenging to achieve. However, the necessity of raising the resonance above this range is less critical due to the potential for cooling, which can reduce its amplitude, possibly bringing it down to the sensitivity level.

Finally, this technical report delves into identifying potential locations for the installation of the components of an active control system for damping and cooling the resonant modes. As discussed in Section 5, by means of a modal analysis, for each mode we obtain the relative displacements at different points of the beam, leading us to conclude that the top of the beam would be the best location to damp the internal resonances of our system within the frequency range of interest.

References

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